

# Global Existence of Solution for a Nonlinear Size-structured Population Model with Distributed Delay in the Recruitment\*

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## Abstract

In this paper we study a nonlinear size-structured population model with distributed delay in the recruitment. The delayed problem is reduced into an abstract initial value problem of an ordinary differential equation in the Banach space by using the delay semigroup techniques. The local existence and uniqueness of solution as well as the continuous dependence on initial conditions are obtained by using the general theory of quasi-linear evolution equations in nonreflexive Banach spaces, while the global existence of solution is obtained by the estimates of the solution and the extension theorem.

**Key words and phrases:** Size-structured populations; distributed delay; global existence

**AMS subject classifications:** 35L02, 35P99.

## 1 Introduction

In this paper, we study the following size-structured population model with the distributed delay:

$$\begin{cases} \frac{\partial n}{\partial t} + \frac{\partial(\gamma(x, N[n(t)](x))n}{\partial x} = -\mu(x, N[n(t)](x))n + \int_{-\tau}^0 R[n(t+\sigma)](x)d\sigma \\ \quad \text{for } x \in [0, \infty), t \geq 0, \\ \gamma(0, N[n(t)](0))n(t, 0) = 0 \quad \text{for } t \geq 0, \\ n(\sigma, x) = \hat{n}(\sigma, x) \quad \text{for } \sigma \in [-\tau, 0], x \in [0, \infty). \end{cases} \quad (1.1)$$

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Here the unknown function  $n(t, x)$  denotes the density of individuals of size  $x \in [0, \infty)$  at time  $t \in [0, \infty)$ ,  $N[n(t)](x)$  is the *environment* or the *interaction variable* (see[1]) experienced by an individual of size  $x$  when the population density is  $n(t, y)$ ,  $\gamma(x, N)$  and  $\mu(x, N)$  is the growth rate and the mortality rate of an individual of size  $x$  when the environment or the interaction variable is  $N$ . We assume that there is a time lag  $-\sigma$  in the process of the recruitment, and  $\sigma \in [-\tau, 0]$ , where  $\tau > 0$  is a constant denoting the maximal time lag. Moreover precisely,  $R[n(t + \sigma)](x)$  is the portion of the recruitment of the new individuals of size  $x$  at time  $t$  which come from the individuals of size  $y \in [0, \infty)$  at time  $t + \sigma$ ,  $\int_{-\tau}^0 R[n(t + \sigma)](x)$  is the entire recruitment of the new individuals of size  $x$  at time  $t$ . Besides,  $\hat{n}(\sigma, x)$  are a given function defined in  $[-\tau, 0] \times [0, \infty)$ . Later on we shall denote

$$\hat{n}_0(x) = \hat{n}(0, x) \quad \text{for } x \in [0, \infty). \quad (1.2)$$

The nonlinear size-structured population model without distributed delay has been studied in [1] and [2]. Global existence of solution has been obtained in [1] and the asymptotic behavior has been studied in [2]. In the model (1.1), we consider not only the environment but also the distributed delay in the recruitment. More precisely, the distributed delay in the recruitment is given by the time lag between conception and birth or laying and hatching of the parasite eggs (see [3]). Moreover, unlike the non-distributed delay case, the time lag considered here can change from 0 to  $\tau$ , i.e., it is distributed in the interval  $[0, \tau]$ . The linear age/size-structured population models with delay in the birth process were studied in [4], [5] and [6]. Recently, some different nonlinear age/size-structured population models with the distributed delay were studied in [7] and [8].

As an extension and development of the work in [1], we reduce the distributed delayed problem (1.1) into an abstract initial value problem of an ordinary differential equation in the Banach space and obtain the global existence of the solution by using the delay semigroup techniques, the general theory of quasi-linear evolution equation in nonreflexive Banach spaces, the estimates of the solution and the extension theorem.

For a given  $K > 0$ , let  $\Omega := [0, \infty) \times [0, K]$ . Then, throughout this paper,  $N$ ,  $\mu(x, N)$ ,  $\gamma(x, N)$ , and  $R$  are supposed to satisfy the following conditions:

(A.1) Let  $M_n := \{v \in L^\infty(0, \infty), v' \in W^{n,1}(0, \infty)\}$ . Then, the operator  $N := L^1 \rightarrow M_0$  and  $N := W^{1,1} \rightarrow M_1$ , satisfies  $N[0] = 0$ , and it is Lipschitzian in the norms  $\|\cdot\|_0 = \|\cdot\|_\infty + \|D(\cdot)\|_{L^1}$  and  $\|\cdot\|_1 = \|\cdot\|_\infty + \|D(\cdot)\|_{W^{1,1}}$ .

(A.2)  $\mu(x, N)$  is a non-negative  $\mathcal{C}^1$ -function, and  $\mu$ ,  $\mu_x$  and  $\mu_N$  are uniformly bounded by  $\mu^0$ ,  $\mu_x^0$  and  $\mu_N^0$ , respectively, for all  $(x, N) \in \Omega$ . Moreover, its partial derivatives  $\mu_x$  and  $\mu_N$  are Lipschitzian functions with respect to  $N$ .

(A.3)  $\gamma(x, N)$  is a strictly positive  $\mathcal{C}^2$ -function for all  $x, N \geq 0$ , upper bounded by  $\gamma^0 > 0$  and lower bounded by  $\gamma_0 > 0$  for all  $(x, N) \in \Omega$ . Moreover, for all  $(x, N)$ ,  $|\gamma_x|$  and  $|\gamma_N|$  are upper bounded by  $\gamma_1^0$ ,  $|\gamma_{xx}|$ ,  $|\gamma_{xN}|$  and  $|\gamma_{NN}|$  are upper bounded by  $\gamma_2^0$ . Finally,  $\gamma_{xx}(x, N)$ ,  $\gamma_{xN}(x, N)$  and  $\gamma_{NN}(x, N)$  are Lipschitzian functions with respect to  $N$ .

(A.4) The positive operator  $R : L^1([-\tau, 0], L^1(0, \infty)) \rightarrow L^1([-\tau, 0], W^{1,1}(0, \infty))$  and  $R : L^1([-\tau, 0], L^1(0, \infty)) \rightarrow L^1([-\tau, 0], L^\infty(0, \infty))$  satisfies  $R[0] = 0$ . For

$\tilde{u} \in L^1([-\tau, 0], L^1(0, \infty))$ ,

$$\left\| \int_{-\tau}^0 R[\tilde{u}(\sigma)] d\sigma \right\|_{L^1(0, \infty)} \leq R_0 \int_0^\infty \int_{-\tau}^0 \tilde{u}(\sigma, y) d\sigma dy,$$

$$\left\| \int_{-\tau}^0 R[\tilde{u}(\sigma)] d\sigma \right\|_{W^{1,1}(0, \infty)} \leq R_1 \int_0^\infty \int_{-\tau}^0 \tilde{u}(\sigma) d\sigma dy,$$

and

$$\left\| \int_{-\tau}^0 R[\tilde{u}(\sigma)] d\sigma \right\|_\infty \leq R_2 \int_0^\infty \int_{-\tau}^0 \tilde{u}(\sigma, y) d\sigma dy.$$

Moreover, for  $\tilde{u}_1, \tilde{u}_2 \in E_1$ , where  $E_1$  is an arbitrary bounded subset in  $L^1([-\tau, 0], L^1(0, \infty))$ ,

$$\left\| \int_{-\tau}^0 R[\tilde{u}_1(\sigma)] d\sigma - \int_{-\tau}^0 R[\tilde{u}_2(\sigma)] d\sigma \right\|_{L^1(0, \infty)} \leq L_R \int_0^\infty \int_{-\tau}^0 |\tilde{u}_1(\sigma, y) - \tilde{u}_2(\sigma, y)| d\sigma dy,$$

and

$$\left\| \int_{-\tau}^0 R[\tilde{u}_1(\sigma)] d\sigma - \int_{-\tau}^0 R[\tilde{u}_2(\sigma)] d\sigma \right\|_{W^{1,1}(0, \infty)} \leq L_{R_x} \int_0^\infty \int_{-\tau}^0 |\tilde{u}_1(\sigma, y) - \tilde{u}_2(\sigma, y)| d\sigma dy.$$

The layout of the rest part is as follows. In Section 2 we reduce the model (1.1) into an abstract initial value problem of an ordinary differential equation in the Banach space. In Section 3 we shall prove the local existence and uniqueness of solution of the the model (1.1) by using the general theory of quasi-linear evolution equations in nonreflexive Banach spaces. In Section 4 we obtain the continuous dependence on initial conditions and the positivity of solutions. In Section 5 we obtain the global existence of solution by the estimates of the solution and the extension theorem. In Section 6 we give the typical examples of the operators  $N$  and  $R$ .

## 2 Reduction

In this section we reduce the problem (1.1) into an abstract Cauchy problem. We refer the reader to see [4], [5] and [9] for similar reductions.

First, we introduce the following Banach spaces:

$$X := L^1(0, \infty), \quad \text{with norm } \|u\|_X = \int_0^\infty |u(x)| dx,$$

$$Y := \left\{ u \in W^{1,1}(0, \infty) : u(0) = 0 \right\}, \quad \text{with norm } \|u\|_Y = \int_0^\infty |u(x)| dx + \int_0^\infty |u'(x)| dx,$$

$$E := L^1([-\tau, 0], X), \quad \text{with norm } \|u\|_E = \int_0^\infty \int_{-\tau}^0 |u(\sigma, x)| d\sigma dx.$$

For a given  $v \in X$ , let  $A(v) : Y \rightarrow X$  be the following linear operator:

$$A(v)u := (\gamma(\cdot, N^v)u)', \quad \text{for } u \in Y, \tag{2.1}$$

where  $N^v := N[v]$ . It is obvious that for a given  $v \in X$ ,  $A(v) \in \mathcal{L}(Y, X)$ . We denote by  $f_1 : X \rightarrow X$  and  $f_2 : E \rightarrow X$ , respectively, the following nonlinear operators:

$$\begin{aligned} f_1(u) &:= -\mu(\cdot, N^u)u, \quad \text{for } u \in X, \\ f_2(\tilde{u}) &:= \int_{-\tau}^0 R[\tilde{u}(\sigma)]d\sigma, \quad \text{for } \tilde{u} \in E, \end{aligned}$$

where  $N^u := N[u]$ .

Using these notations, we rewrite the model (1.1) into the following abstract initial value problem for a retarded differential equation in the Banach space  $X$ :

$$\begin{cases} \frac{dn(t)}{dt} + A(n(t))n(t) = f_1(n(t)) + f_2(n_t), & t \geq 0 \\ n(0) = \hat{n}_0, \\ n_0 = \hat{n}, \end{cases} \quad (2.2)$$

where  $n : [0, +\infty) \rightarrow X$  is defined as  $n(t) := n(t, \cdot)$  and  $n_t : [-\tau, 0] \rightarrow X$  is defined as  $n_t(\sigma) := n(t + \sigma)$ ,  $\sigma \in [-\tau, 0]$ .

Next, we introduce the following operators in the Banach space  $E$ :

$$\begin{aligned} (G\tilde{u})(\sigma) &:= -\frac{d}{d\sigma}\tilde{u}, \quad \text{with domain } D(G) = W^{1,1}([-\tau, 0], X), \\ Q\tilde{u} &:= \tilde{u}(0), \quad \text{for } \tilde{u} \in D(G). \end{aligned}$$

We note that  $G \in \mathcal{L}(D(G), E)$  and  $Q \in \mathcal{L}(D(G), X)$ . We now let

$$\mathbb{X} := E \times X, \quad \text{with norm } \|(\tilde{u}, u)\|_{\mathbb{X}} = \|\tilde{u}\|_E + \|u\|_X$$

and

$$\begin{aligned} \mathbb{Y} &:= \left\{ (\tilde{u}, u) \in W^{1,1}([-\tau, 0], X) \times W^{1,1}(0, \infty), Q\tilde{u} = u, u(0) = 0 \right\}, \\ &\text{with norm } \|(\tilde{u}, u)\|_{\mathbb{Y}} = \|\tilde{u}\|_E + \|\tilde{u}'\|_E + \|u\|_Y, \end{aligned}$$

where  $\tilde{u}'_1(\sigma, x) = \frac{\partial \tilde{u}(\sigma, x)}{\partial \sigma}$ . For a given  $\mathbf{V} = (\tilde{v}, v) \in \mathbb{X}$ , let  $\mathbf{A}(\mathbf{V}) : \mathbb{Y} \rightarrow \mathbb{X}$  be the following operator:

$$\mathbf{A}(\mathbf{V})\mathbf{U} := \begin{pmatrix} G & 0 \\ 0 & A(v) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ u \end{pmatrix}, \quad \text{for } \mathbf{U} = \begin{pmatrix} \tilde{u} \\ u \end{pmatrix} \in \mathbb{Y}. \quad (2.3)$$

It is obvious that for given  $\mathbf{V} = (\tilde{v}, v) \in \mathbb{X}$ ,  $\mathbf{A}(\mathbf{V}) \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ . We also denote by  $\mathbf{F} : \mathbb{X} \rightarrow \mathbb{X}$ , the following nonlinear operator:

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} 0 \\ f_1(u) + f_2(\tilde{u}) \end{pmatrix}, \quad \text{for } \mathbf{U} = \begin{pmatrix} \tilde{u} \\ u \end{pmatrix} \in \mathbb{X}. \quad (2.4)$$

Using these notations, we see that the problem (2.1) can be equivalently rewrite into the following abstract initial value problem of an ordinary differential equation in the Banach space  $\mathbb{X}$ :

$$\begin{cases} \mathbf{U}'(t) + \mathbf{A}(\mathbf{U}(t))\mathbf{U}(t) = \mathbf{F}(\mathbf{U}(t)), & t > 0, \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \quad (2.5)$$

where  $\mathbf{U}(t) = \begin{pmatrix} n_t \\ n(t) \end{pmatrix}$  and  $\mathbf{U}_0 = \begin{pmatrix} \hat{n} \\ \hat{n}_0 \end{pmatrix}$ , where  $\hat{n}_0$  is the function defined in (1.2).

To describe the relationship between the problem (2.2) and the problem (2.5), we write down the following preliminary result:

**Lemma 2.1** *If for any initial condition  $(\hat{n}, \hat{n}_0) \in W^{1,1}([-\tau, 0], X) \times Y$ , there exists a time  $T > 0$  such that the problem (2.2) has a unique solution  $n \in C([-\tau, T], W^{1,1}(0, \infty)) \cap C([0, T], Y) \cap C^1([0, T], X)$ , then the problem (2.5) has a unique solution  $\mathbf{U}(t) = \begin{pmatrix} n_t \\ n(t) \end{pmatrix}$  and  $\mathbf{U} \in C([0, T], \mathbb{Y}) \cap C^1([0, T], \mathbb{X})$ . Conversely, if for any initial condition  $\mathbf{U}_0 \in \mathbb{Y}$ , there exists a time  $T > 0$  such that (2.5) has a unique solution  $\mathbf{U} \in C([0, T], \mathbb{Y}) \cap C^1([0, T], \mathbb{X})$ , then  $\mathbf{U}$  has the form  $\mathbf{U}(t) = \begin{pmatrix} n_t \\ n(t) \end{pmatrix}$  for all  $t \in [0, T]$ , and by extending its second component  $n = n(t)$  to  $[-\tau, T]$  such that  $n(t) = \hat{n}$  for  $t \in [-\tau, 0)$ , we have that a unique solution of the problem (2.2).*

*Proof:* We only need to prove that for any initial condition  $\mathbf{U}_0 \in \mathbb{Y}$ , there exists a time  $T > 0$  such that the problem (2.5) has a unique solution  $\mathbf{U} \in C([0, T], \mathbb{Y}) \cap C^1([0, T], \mathbb{X})$ , then  $\mathbf{U}$  has the form  $\mathbf{U}(t) = \begin{pmatrix} n_t \\ n(t) \end{pmatrix}$  for all  $t \in [0, T]$ . To see this we assume that

$\mathbf{U}(t) = \begin{pmatrix} U(t) \\ n(t) \end{pmatrix}$  (for  $t \in [0, T]$ ). Since  $\mathbf{U} \in C([0, T], \mathbb{Y})$ , we have  $QU(t) = n(t)$  for all  $t \in [0, T]$ , i.e.,  $U(t)(0) = n(t)$  for all  $t \in [0, T]$  (recall that  $U(t) \in W^{1,1}([-\tau, 0], X)$  for every  $t \in [0, T]$ ). Let  $v(t, \sigma) = U(t)(\sigma)$  (for  $t \in [0, T]$  and  $\sigma \in [-\tau, 0]$ ). Then from the equation satisfied by  $U$  we see that  $v$  satisfies the equation  $\partial v / \partial t - \partial v / \partial \sigma = 0$ , so that it is a leftward traveling wave, i.e., it has the form  $v(t, \sigma) = g(t + \sigma)$  for some function  $g = g(s)$  defined for  $s \in [-\tau, T]$ . For  $s \in [0, T]$  we have  $g(s) = v(s, 0) = U(s)(0) = n(s)$ . Hence, for  $t + \sigma \in [0, T]$  we have

$$U(t)(\sigma) = v(t, \sigma) = g(t + \sigma) = n(t + \sigma) = n_t(\sigma).$$

Moreover, since  $U(0)(\sigma) = \hat{n}(\sigma, \cdot)$  and  $U(t)(\sigma) = v(t, \sigma) = g(t + \sigma)$  for  $-\tau \leq t + \sigma < 0$ , we see that  $g(s) = \hat{n}(s, \cdot)$  for  $s \in [-\tau, 0)$  and  $U(t)(\sigma) = \hat{n}(t + \sigma, \cdot)$  for  $t + \sigma \in [-\tau, 0)$ . Hence, by defining  $n(t) = \hat{n}(t, \cdot)$  for  $t \in [-\tau, 0)$ , we see that  $U(t)(\sigma) = n_t(\sigma)$  also holds for all  $t \in [0, T]$  and  $\sigma \in [-\tau, 0]$ . This proves the desired assertion.  $\square$

### 3 Local existence and uniqueness of solution

In this section we shall prove the local existence and uniqueness of solution for the problem (2.5) by using the general theory of quasi-linear evolution equations in nonreflexive Banach spaces(see[1] and [10]). For this purpose, we shall verify the hypotheses in the Theorems which are proposed and proven by Kobayasi and Sanekata(see [10] and Theorem I and Theorem II of [1]), and apply these theorems to the problem (2.5). In the sequel we denote by  $\mathbb{W}$  the open subset of  $\mathbb{Y}$  which contained in the closed ball in  $\mathbb{Y}$  with center 0 and radius  $r > \|\mathbf{U}_0\|_{\mathbb{Y}}$ ,  $S(\mathbb{X}, M, \alpha)$  the set of all stable families of negative generators of  $C^0$ -semigroups in  $\mathbb{X}$  with stability index  $(M, \alpha)$ , and  $\mathcal{B}(X, Y)$  the set of all bounded linear operators for the real Banach space  $X$  to the real Banach space  $Y$  ( $\mathcal{B}(X) := \mathcal{B}(X, X)$ ).

**Lemma 3.1** *The space  $\mathbb{Y}$  is densely and continuously embedded in  $\mathbb{X}$ . There is an isomorphism  $\mathbf{S}$  of  $\mathbb{Y}$  onto  $\mathbb{X}$ . Moreover, there exists two positive constants  $c_1$  and  $c_2$  such that*

$$c_1 \|\mathbf{U}\|_{\mathbb{Y}} \leq \|\mathbf{S}\mathbf{U}\|_{\mathbb{X}} \leq c_2 \|\mathbf{U}\|_{\mathbb{Y}}.$$

*Proof:* Since there exist a generator with domain  $\mathbb{Y}$  which generates a strongly continuous semigroup on  $\mathbb{X}$  (see Proposition 4.4 of [5]), by Corollary 2.5 in Chapter 1 of [15], the first assertion follows.

The isomorphism  $\mathbf{S} : \mathbb{Y} \rightarrow \mathbb{X}$  is

$$\mathbf{S} \begin{pmatrix} \tilde{u} \\ u \end{pmatrix} = \begin{pmatrix} -\tilde{u}'_1 + \tilde{u} \\ u' + u \end{pmatrix}, \quad \text{for } \begin{pmatrix} \tilde{u} \\ u \end{pmatrix} \in \mathbb{Y}.$$

$\mathbf{S}$  is bijective because for every  $\mathbf{F} = (\tilde{f}, f) \in \mathbb{X}$ , the equations

$$\begin{cases} -\frac{\partial}{\partial \sigma} \tilde{u}(\sigma, x) + \tilde{u}(\sigma, x) = \tilde{f}(\sigma, x), & -\tau < \sigma < 0, 0 < x < \infty, \\ \frac{du(x)}{dx} + u(x) = f(x), & 0 < x < \infty, \\ \tilde{u}(0, x) = u(x), & 0 < x < \infty, \\ u(0) = 0, \end{cases} \quad (3.1)$$

have a unique solution, given by

$$\begin{cases} \tilde{u}(\sigma, x) = e^{\sigma} u(x) + e^{\sigma} \int_{\sigma}^0 e^{-\xi} \tilde{f}(\xi, x) d\xi, \\ u(x) = e^{-x} \int_0^x e^s f(s) ds, \end{cases} \quad (3.2)$$

and the unique solution  $(\tilde{u}, u) \in \mathbb{Y}$ .

Let  $\mathbf{U} = (\tilde{u}, u) \in \mathbb{Y}$ . It is easy to see that  $\|\mathbf{U}\|_{\mathbb{Y}} \geq \|\mathbf{S}\mathbf{U}\|_{\mathbb{X}}$ . This implies that  $c_2 = 1$ . Since

$$\int_0^{\infty} |u(x)| dx \leq \int_0^{\infty} e^{-x} \int_0^x e^s |f(s)| ds dx \leq \int_0^{\infty} \left( \int_s^{\infty} e^{-x} dx \right) e^s |f(s)| ds = \int_0^{\infty} |f(x)| dx$$

and

$$\begin{aligned}
& \int_0^\infty \int_{-\tau}^0 |\tilde{u}(\sigma, x)| d\sigma dx \\
& \leq \int_0^\infty \int_{-\tau}^0 |e^\sigma u(x)| d\sigma dx + \int_0^\infty \int_{-\tau}^0 \left| e^\sigma \int_\sigma^0 e^{-\xi} \tilde{f}(\xi, x) d\xi \right| d\sigma dx \\
& \leq \tau \int_0^\infty |u(x)| dx + \int_0^\infty \int_{-\tau}^0 \int_\sigma^0 e^{\sigma-\xi} |\tilde{f}(\xi, x)| d\xi d\sigma dx \\
& \leq \tau \int_0^\infty |f(x)| dx + \tau \int_0^\infty \int_{-\tau}^0 |\tilde{f}(\sigma, x)| d\sigma dx,
\end{aligned}$$

we have

$$\begin{aligned}
\int_0^\infty |u'(x)| dx &= \int_0^\infty |f(x) - u(x)| dx \\
&\leq \int_0^\infty |u(x)| dx + \int_0^\infty |f(x)| dx \\
&\leq 2 \int_0^\infty |f(x)| dx
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \int_{-\tau}^0 \left| \frac{\partial \tilde{u}(\sigma, x)}{\partial \sigma} \right| d\sigma dx &= \int_0^\infty \int_{-\tau}^0 |\tilde{u}(\sigma, x) - \tilde{f}(\sigma, x)| d\sigma dx \\
&\leq \int_0^\infty \int_{-\tau}^0 |\tilde{u}(\sigma, x)| d\sigma dx + \int_0^\infty \int_{-\tau}^0 |\tilde{f}(\sigma, x)| d\sigma dx \\
&\leq \tau \int_0^\infty |f(x)| dx + (\tau + 1) \int_0^\infty \int_{-\tau}^0 |\tilde{f}(\sigma, x)| d\sigma dx.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathbf{U}\|_{\mathbb{Y}} &= \int_0^\infty \int_{-\tau}^0 |\tilde{u}(\sigma, x)| d\sigma dx + \int_0^\infty \int_{-\tau}^0 \left| \frac{\partial \tilde{u}(\sigma, x)}{\partial \sigma} \right| d\sigma dx + \int_0^\infty |u(x)| dx + \int_0^\infty |u'(x)| dx \\
&\leq (2\tau + 1) \int_0^\infty \int_{-\tau}^0 |\tilde{f}(\sigma, x)| d\sigma dx + (2\tau + 3) \int_0^\infty |f(x)| dx \\
&\leq (2\tau + 3) \|\mathbf{S}\mathbf{U}\|_{\mathbb{X}}.
\end{aligned}$$

This implies that  $c_2 = \frac{1}{2\tau + 3}$ . This completes the proof.  $\square$

**Lemma 3.2** *For each  $\mathbf{W} \in \mathbb{W}$ ,  $\mathbf{A}(\mathbf{W})$  is a linear operator in  $\mathbb{W}$ . Let  $T$  be a positive constant. For each  $\rho \geq 0$ , there exist two constants  $M \geq 1$  and  $\alpha \geq 0$  such that*

$$(\mathbf{A}(\mathbf{V}(t)))_{t \in [0, T]} \in S(\mathbb{X}, M, \alpha), \quad \forall \mathbf{V}(t) \in \mathcal{D}_\rho,$$

where  $\mathcal{D}_\rho := \{\mathbf{V}(t) \in C([0, T], \mathbb{W}) : \|\mathbf{V}(t) - \mathbf{V}(s)\|_{\mathbb{X}} \leq \rho|t - s|, 0 \leq s < t \leq T\}$ .

*Proof:* From (2.1) and (2.3), the first assertion follows. We denote by  $\mathbf{I} = \begin{pmatrix} \tilde{I} & 0 \\ 0 & I \end{pmatrix}$  the identity operator in  $\mathbb{X}$ , where  $\tilde{I}$  and  $I$  represent the identity operators in  $E$  and  $X$ ,

respectively. For each  $\mathbf{W} \in \mathbb{W}$ , we introduce the operator  $\mathbf{A}_1(\mathbf{W}) := \mathbf{A}(\mathbf{W}) + \mathbf{I}$ . Then for each  $\mathbf{W} \in \mathbb{W}$ ,  $\mathbf{A}(\mathbf{W}) = \mathbf{A}_1(\mathbf{W}) - \mathbf{I}$ . Since  $\|\mathbf{I}\|_{\mathcal{L}(\mathbb{X})} \leq 1$ , by Theorem 2.3 in Chapter 5 of [15], the second assertion follows if we prove the stability of the family  $(\mathbf{A}_1(\mathbf{V}(t)))_{t \in [0, T]}$ . Since for a given  $\mathbf{W} \in \mathbb{W}$ , the operator  $\mathbf{A}_1(\mathbf{W})$  does not depend on  $t$  we can take  $T = \infty$  and, in order to prove the stability of the family  $(\mathbf{A}_1(\mathbf{V}(t)))_{t \in [0, T]}$  we only need to show that for each  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ ,  $-\mathbf{A}_1(\mathbf{W})$  generates a contraction semigroup. To this end we prove that for each  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ ,  $-\mathbf{A}_1(\mathbf{W})$  satisfy the conditions of Hille-Yosida Theorem (see Theorem 3.1 in Chapter 1 of [15]). Since for each  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ ,  $-\mathbf{A}_1(\mathbf{W})$  is closed,  $\overline{D(-\mathbf{A}_1(\mathbf{W}))} = \overline{\mathbb{Y}} = \mathbb{X}$  and the resolvent set  $\rho(-\mathbf{A}_1(\mathbf{W}))$  of  $-\mathbf{A}_1(\mathbf{W})$  contains  $\mathbb{R}^+$  (see Proposition 2.1 of [11] and Proposition 3.2 of [5]), we only need to show that for every  $\lambda > 0$ ,  $\|R(\lambda, -\mathbf{A}_1(\mathbf{W}))\| \leq \frac{1}{\lambda}$ . For  $\mathbf{F} \in \mathbb{X}$ , let  $U = R(\lambda, -\mathbf{A}_1(\mathbf{W}))\mathbf{F}$ . Then  $\mathbf{U}$  satisfies the equation

$$(\lambda \mathbf{I} + \mathbf{A}_1(\mathbf{W}))\mathbf{U} = \mathbf{F}. \quad (3.3)$$

By writing  $\mathbf{U} = (\tilde{u}(\sigma, x), u(x))$  and  $\mathbf{F} = (\tilde{f}(\sigma, x), f(x))$ , we see that the above equation can be rewritten as follows:

$$\begin{cases} \lambda \tilde{u}(\sigma, x) + \tilde{u}(\sigma, x) - \frac{\partial}{\partial \sigma} \tilde{u}(\sigma, x) = \tilde{f}(\sigma, x), & -\tau < \sigma < 0, 0 < x < m, \\ \lambda u(x) + u(x) + \frac{d}{dx}(\gamma(x, N^w)u(x)) = f(x), & 0 < x < m, \\ U(0, x) = u(x), & 0 < x < m, \\ u(0) = 0. \end{cases} \quad (3.4)$$

Then we have that

$$\tilde{u}(\sigma, x) = e^{(\lambda+1)\sigma} u(x) + e^{(\lambda+1)\sigma} \int_{\sigma}^0 e^{-(\lambda+1)\xi} \tilde{f}(\xi, x) d\xi, \quad (3.5)$$

and

$$u(x) = E_{\lambda}(x) \int_0^x (E_{\lambda}(s) \gamma(s, N^w))^{-1} f(s) ds, \quad (3.6)$$

where  $E_{\lambda}(x) = \exp \left\{ - \int_0^x \frac{\lambda + 1 + \gamma'(s, N^w)}{\gamma(s, N^w)} ds \right\}$ . We deduce an useful expression of  $R(\lambda, -\mathbf{A}_1(\mathbf{W}))$ . From (3.4), (3.5) and (3.6), we have that

$$R(\lambda, -\mathbf{A}_1(\mathbf{W})) = \begin{pmatrix} R(\lambda, -G_0) & \varepsilon_{\lambda} R(\lambda, -A_1(w)) \\ 0 & R(\lambda, -A_1(w)) \end{pmatrix}, \quad (3.7)$$

where  $\varepsilon_{\lambda} := e^{(\lambda+1)\sigma}$  for  $\sigma \in [-\tau, 0)$ ,  $A_1(w) := A(w) + I$  and  $G_0$  is the following operator in the Banach space  $E = L^1([-\tau, 0], X)$ :

$$(G_0 \tilde{u})(\sigma) := -\frac{d}{d\sigma} \tilde{u} + \tilde{u}, \quad \text{with domain } D(G_0) = \left\{ \tilde{u} \in W^{1,1}([-\tau, 0], X) : \tilde{u}(0, x) = 0 \right\}.$$



Since  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ , there exists a bounded set  $W$  contained in  $Y$  such that  $w \in W$ . By the proof of H2 in [1], we have that for each  $w \in W$ ,  $-A(w)$  generates a contraction semigroup  $\{T(t)\}_{t \geq 0}$ . Since  $-A_1(w) = -A(w) - I$ , by the theory of the rescaled semigroups (see Example II 2.2 of [12]), we have that for each  $w \in W$ ,  $-A_1(w)$  generates a semigroup  $\{S(t)\}_{t \geq 0}$  such that  $S(t) = e^{-t}T(t)$  for  $t \geq 0$ . Then  $\|S(t)\| \leq e^{-t}$  for  $t \geq 0$ . By Corollary 3.8 in Chapter 1 of [15], we have that for every  $\lambda > 0$ ,  $\|R(\lambda, -A_1(w))\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda+1}$ . Hence, for  $\mathbf{F} = (\tilde{f}(\sigma, x), f(x)) \in \mathbb{X}$  and  $\lambda > 0$ ,

$$\begin{aligned}
& \|R(\lambda, -\mathbf{A}_1(\mathbf{W}))\mathbf{F}\|_{\mathbb{X}} \\
&= \|R(\lambda, G_0)F + \varepsilon_\lambda R(\lambda, -A_1(w))f\|_E + \|R(\lambda, -A_1(w))f\|_X \\
&\leq \int_0^\infty \int_{-\tau}^0 e^{(\lambda+1)\sigma} \int_\sigma^0 e^{-(\lambda+1)\xi} |\tilde{f}(\xi, x)| d\xi d\sigma dx \\
&\quad + \int_0^\infty \int_{-\tau}^0 e^{(\lambda+1)\sigma} |R(\lambda, -A_1(w))f(x)| d\sigma dx + \frac{1}{\lambda+1} \|f\|_X \\
&\leq \int_0^\infty \int_{-\tau}^0 e^{-(\lambda+1)\xi} |\tilde{f}(\xi, x)| \int_{-\tau}^\xi e^{(\lambda+1)\sigma} d\sigma d\xi dx \\
&\quad + \int_{-\tau}^0 e^{(\lambda+1)\sigma} d\sigma \int_0^\infty |R(\lambda, -A_1(w))f(x)| dx + \frac{1}{\lambda+1} \|f\|_X \\
&\leq \frac{1}{\lambda+1} \|\tilde{f}\|_E + \frac{1}{(\lambda+1)^2} \|f\|_X + \frac{1}{\lambda+1} \|f\|_X \\
&\leq \frac{1}{\lambda+1} \|\tilde{f}\|_E + \left( \frac{1}{(\lambda+1)^2} + \frac{1}{\lambda+1} \right) \|f\|_X \\
&\leq \frac{1}{\lambda} \|\mathbf{F}\|_{\mathbb{X}}.
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 3.3** *For any  $\mathbf{W} \in \mathbb{W}$ , there exists an operator  $\mathbf{B}(\mathbf{W})$  such that*

$$\mathbf{S}\mathbf{A}(\mathbf{W})\mathbf{S}^{-1} = \mathbf{A}(\mathbf{W}) + \mathbf{B}(\mathbf{W}), \mathbf{W} \in \mathbb{W},$$

where  $\mathbf{S}$  is the isomorphism defined in Lemma 3.1. Moreover, there exist two positive numbers  $\lambda_{\mathbf{B}}$  and  $\mu_{\mathbf{B}}$  such that

$$\|\mathbf{B}(\mathbf{W})\|_{\mathbb{X}} \leq \lambda_{\mathbf{B}}, \text{ for } \mathbf{W} \in \mathbb{W}, \quad (3.8)$$

and

$$\|\mathbf{B}(\mathbf{W}_1) - \mathbf{B}(\mathbf{W}_2)\|_{\mathbb{X}} \leq \mu_{\mathbf{B}} \|\mathbf{W}_1 - \mathbf{W}_2\|_{\mathbb{Y}}, \text{ for } \mathbf{W}_1, \mathbf{W}_2 \in \mathbb{W}. \quad (3.9)$$

*Proof:* For a given  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ , we have that

$$\mathbf{B}(\mathbf{W})\mathbf{U} = (\mathbf{S}\mathbf{A}(\mathbf{W}) - \mathbf{A}(\mathbf{W})\mathbf{S})\mathbf{S}^{-1}\mathbf{U}, \text{ for } \mathbf{U} \in \mathbb{X}.$$

To find the concrete expression of  $\mathbf{B}(\mathbf{W})$ , we compute the "commutator"  $\mathbf{S}\mathbf{A}(\mathbf{W}) - \mathbf{A}(\mathbf{W})\mathbf{S}$ :

$$(\mathbf{S}\mathbf{A}(\mathbf{W}) - \mathbf{A}(\mathbf{W})\mathbf{S})\mathbf{U} = \begin{pmatrix} 0 \\ D^2(\gamma(\cdot, N^w))u + D(\gamma(\cdot, N^w))u' \end{pmatrix}, \text{ for } \mathbf{U} = \begin{pmatrix} \tilde{u} \\ u \end{pmatrix} \in \mathbb{Y}$$

where  $D(\gamma(x, N^w)) = \gamma'_1(x, N^w) + \gamma'_2(x, N^w)\frac{dN^w}{dx}$ .

Let  $\mathbf{U} \in \mathbb{X}$  and  $\mathbf{V} = \mathbf{S}^{-1}\mathbf{U} \in \mathbb{Y}$ . Then,

$$\mathbf{B}(\mathbf{W})\mathbf{U} = (\mathbf{S}\mathbf{A}(\mathbf{W}) - \mathbf{A}(\mathbf{W})\mathbf{S})\mathbf{S}^{-1}\mathbf{U} = (\mathbf{S}\mathbf{A}(\mathbf{W}) - \mathbf{A}(\mathbf{W})\mathbf{S})\mathbf{V}$$

The first component of  $\mathbf{B}(\mathbf{W})$  is zero. The second component of  $\mathbf{B}(\mathbf{W})$  is similar as the operator  $B(w)$  in the proof of H3 in [1]. Under (A.1) and (A.3), we can obtain (3.8) and (3.9) by using the same method. This completes the proof.  $\square$

*Remark 3.1:* Since for a given  $\mathbf{W} \in \mathbb{W}$ , the operator  $\mathbf{B}(\mathbf{W})$  does not depend on  $t$ ,  $\mathbf{B}(\mathbf{W}) : [0, T_0] \rightarrow \mathcal{B}(\mathbb{X})$  is a strongly measurable operator valued function with  $T_0 = \infty$ .

**Lemma 3.4** *For each  $\mathbf{W} \in \mathbb{W}$ ,  $D(\mathbf{A}(\mathbf{W})) \supset \mathbb{Y}$  and  $\mathbf{A}(\mathbf{W}) \in \mathcal{B}(\mathbb{Y}, \mathbb{X})$ . Moreover, there exists a constant  $\mu_{\mathbf{A}}$  such that*

$$\|\mathbf{A}(\mathbf{W}_1) - \mathbf{A}(\mathbf{W}_2)\|_{\mathbb{Y}, \mathbb{X}} \leq \mu_{\mathbf{A}} \|\mathbf{W}_1 - \mathbf{W}_2\|_{\mathbb{X}}, \text{ for } \mathbf{W}_1, \mathbf{W}_2 \in \mathbb{W}. \quad (3.10)$$

*Proof:* For any  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ ,  $D(\mathbf{A}(\mathbf{W})) = \mathbb{Y}$ . Let  $\mathbf{U} \in \mathbb{Y}$ , we have that

$$\begin{aligned} \|\mathbf{A}(\mathbf{W})\mathbf{U}\|_{\mathbb{X}} &= \left\| G\tilde{u} \right\|_E + \|(A(w))u\|_X \\ &\leq \left\| \frac{\partial \tilde{u}(\sigma, x)}{\partial \sigma} \right\|_E + \|(A(w))u\|_X. \end{aligned}$$

Since  $A(w)$  is the same as the one in the proof of H4 in [1], we have that for each  $w \in W$ , there exists a constant  $c > 0$  such that  $\|(A(w))u\|_X \leq c\|u\|_Y$ . Then

$$\|\mathbf{A}(\mathbf{W})\mathbf{U}\|_{\mathbb{X}} \leq (1 + c)\|\mathbf{U}\|_{\mathbb{Y}}.$$

This completes the proof of the first assertion.

For any  $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{W}$  ( $\mathbf{W}_1 = (\tilde{w}_1, w_1)$ ,  $\mathbf{W}_2 = (\tilde{w}_2, w_2)$ ),

$$\|(\mathbf{A}(\mathbf{W}_1) - \mathbf{A}(\mathbf{W}_2))\mathbf{U}\|_{\mathbb{X}} = \|(A(w_1) - A(w_2))u\|_X.$$

Under (A.1) and (A.3), we can obtain (3.10) by using the same method in the proof of H4 in [1]. This completes the proof.  $\square$

*Remark 3.2:* Since for a given  $\mathbf{W} \in \mathbb{W}$ ,  $\mathbf{A}(\mathbf{W})$  is an autonomous first-order differential operator,  $\mathbf{A}(\mathbf{W}) : [0, T_0] \rightarrow \mathcal{B}(\mathbb{Y}, \mathbb{X})$  is strongly continuous with  $T_0 = \infty$ .

**Lemma 3.5** *There exist positive numbers  $\lambda_{\mathbf{F}}$ ,  $\mu_{\mathbf{F}}$  and  $\bar{\mu}_{\mathbf{F}}$  such that*

$$\|\mathbf{F}(\mathbf{W})\|_{\mathbb{Y}} \leq \lambda_{\mathbf{F}}, \text{ for } \mathbf{W} \in \mathbb{W}, \quad (3.11)$$

$$\|\mathbf{F}(\mathbf{W}_1) - \mathbf{F}(\mathbf{W}_2)\|_{\mathbb{X}} \leq \mu_{\mathbf{F}} \|\mathbf{W}_1 - \mathbf{W}_2\|_{\mathbb{X}}, \text{ for } \mathbf{W}_1, \mathbf{W}_2 \in \mathbb{W}, \quad (3.12)$$

$$\|\mathbf{F}(\mathbf{W}_1) - \mathbf{F}(\mathbf{W}_2)\|_{\mathbb{Y}} \leq \bar{\mu}_{\mathbf{F}} \|\mathbf{W}_1 - \mathbf{W}_2\|_{\mathbb{Y}}, \text{ for } \mathbf{W}_1, \mathbf{W}_2 \in \mathbb{W}. \quad (3.13)$$

*Proof:* Let  $\mathbf{W} = (\tilde{w}, w) \in \mathbb{W}$ ,  $\mathbf{W}_1 = (\tilde{w}_1, w_1) \in \mathbb{W}$  and  $\mathbf{W}_2 = (\tilde{w}_2, w_2) \in \mathbb{W}$ . We have that

$$\|\mathbf{F}(\mathbf{W})\|_{\mathbb{Y}} \leq \|f_1(w)\|_Y + \|f_2(\tilde{w})\|_Y, \quad (3.14)$$

$$\|\mathbf{F}(\mathbf{W}_1) - \mathbf{F}(\mathbf{W}_2)\|_{\mathbb{X}} \leq \|f_1(w_1) - f_1(w_2)\|_X + \|f_2(\tilde{w}_1) - f_2(\tilde{w}_2)\|_X, \quad (3.15)$$

and

$$\|\mathbf{F}(\mathbf{W}_1) - \mathbf{F}(\mathbf{W}_2)\|_{\mathbb{Y}} \leq \|f_1(w_1) - f_1(w_2)\|_Y + \|f_2(\tilde{w}_1) - f_2(\tilde{w}_2)\|_Y. \quad (3.16)$$

Under (A.1) and (A.3), by using the same method in the proof of H5 in [1], we have that there exist positive numbers  $\lambda_{f_1}$ ,  $\mu_{f_1}$  and  $\bar{\mu}_{f_1}$  such that

$$\|f_1(w)\|_Y \leq \lambda_{f_1}, \quad (3.17)$$

$$\|f_1(w_1) - f_1(w_2)\|_X \leq \mu_{f_1} \|w_1 - w_2\|_X, \quad (3.18)$$

and

$$\|f_1(w_1) - f_1(w_2)\|_Y \leq \mu_{f_1} \|w_1 - w_2\|_Y. \quad (3.19)$$

Then we only need to consider  $f_2$ . Under (A.4), we have

$$\|f_2(\tilde{w})\|_Y = \left\| \int_{-\tau}^0 R[\tilde{w}(\sigma, y)] d\sigma \right\|_{W^{1,1}(0, \infty)} \leq R_1 \int_0^\infty \int_{-\tau}^0 \tilde{w}(\sigma, y) d\sigma dy \leq \mu_{f_2}, \quad (3.20)$$

$$\|f_2(\tilde{w}_1) - f_2(\tilde{w}_2)\|_Y \leq L_R \|\tilde{w}_1 - \tilde{w}_2\|_E \quad (3.21)$$

and

$$\|f_2(\tilde{w}_1) - f_2(\tilde{w}_2)\|_X \leq L_{R_x} \|\tilde{w}_1 - \tilde{w}_2\|_E \quad (3.22)$$

From (3.14-3.22), we obtain (3.11), (3.12) and (3.13). This completes the proof.  $\square$

*Remark 3.3:* For each  $\mathbf{W} \in \mathbb{W}$ ,  $\mathbf{F}(\mathbf{W})$  is a well defined function belonging to  $\mathbb{Y}$ . Since  $\mathbf{F}(\mathbf{W})$  does not depend on  $t$ , for each  $\mathbf{W} \in \mathbb{W}$ ,  $\mathbf{F}(\cdot, \mathbf{W})$  is continuous in  $\mathbb{X}$  on  $[0, T_0]$  and is strongly measurable in  $\mathbb{Y}$  with  $T_0 = \infty$ .

By Lemma 3.1-3.5 and Remark 3.1-3.3, the hypotheses of Theorem I and II in [1] are fulfilled with  $T_0 = \infty$  and  $\mathbb{W}$  the open subset of  $\mathbb{Y}$  which contained in an arbitrary closed ball in  $\mathbb{Y}$  with center 0 and radius  $r > \|\mathbf{U}_0\|_{\mathbb{Y}}$ . Hence, we have the following result:

**Lemma 3.6** Under (A.1) – (A.4), for any initial condition  $\mathbf{U}_0 \in \mathbb{Y}$ , there exists a time  $T > 0$  such that the problem (2.5) has a unique solution  $\mathbf{U} \in C([0, T], \mathbb{Y}) \cap C^1([0, T], \mathbb{X})$ . Moreover, the family of operators  $\{\mathbb{U}(t, s)\}$ ,  $(t, s) \in \Delta$ , generated by  $\{\mathbf{A}(\mathbf{U}(t))\}_{t \in [0, T]}$  is stable with stability index  $(M, \alpha)$  in  $\mathbb{X}$  and  $(\tilde{M}, \tilde{\alpha})$  in  $\mathbb{Y}$ , where  $(\tilde{M}, \tilde{\alpha}) = (M\|\mathbf{S}\|_{\mathbb{Y}, \mathbb{X}}\|\mathbf{S}^{-1}\|_{\mathbb{X}, \mathbb{Y}}, \lambda_{\mathbf{B}}M + \alpha)$  and it satisfies the properties of Theorem II in [1].

By Lemma 2.1, we have the following result:

**Theorem 3.1** Under (A.1)–(A.4), for any initial condition  $(\hat{n}, \hat{n}_0) \in W^{1,1}([-\tau, 0], X) \times Y$ , there exists a time  $T > 0$  such that (2.2) has a unique solution  $n \in C([-\tau, T], W^{1,1}(0, \infty)) \cap C([0, T], Y) \cap C^1([0, T], X)$ .

## 4 Continuous Dependence on Initial Conditions and Positivity of Solutions

In this section we obtain the continuous dependence on initial conditions and the positivity of solutions.

Since under (A.1) – (A.4), the hypotheses of Theorem I and II in [1] are fulfilled and the evolution operator of (2.5) satisfies Theorem II in [1], we obtain the following results by using the same methods for proving Theorem 2 and Theorem 3 in [1].

**Lemma 4.1** Let  $\mathbf{U}$  and  $\mathbf{V}$  be solutions of the problem (2.5) with initial conditions  $\mathbf{U}_0$  and  $\mathbf{V}_0$ , respectively, in  $\mathbb{Y}$ . Then, under (A.1) – (A.4), for all  $0 < t < T$ , there exists a constant  $\zeta(r, T)$  such that

$$\|\mathbf{U}(t) - \mathbf{V}(t)\|_{\mathbb{X}} \leq Me^{\alpha t} \|\mathbf{U}_0 - \mathbf{V}_0\|_{\mathbb{X}} (1 + t\zeta(r, T)),$$

with  $r > \max\{\|\mathbf{U}_0\|_{\mathbb{Y}}, \|\mathbf{V}_0\|_{\mathbb{Y}}\}$ , where  $T$  is a common local existence time of  $\mathbf{U}$  and  $\mathbf{V}$ .

**Lemma 4.2** If the initial condition  $\mathbf{U}_0 \geq 0$ , then under (A.1) – (A.4), the solution of the problem (2.5) is non-negative for any  $t \in [0, T]$ , where  $T$  is the local existence time of the solution.

By Lemma 2.1, we have the following results:

**Theorem 4.1** Let  $n$  and  $m$  be solutions of (2.2) with initial conditions  $(\hat{n}, \hat{n}_0)$  and  $(\hat{m}, \hat{m}_0)$ , respectively, in  $W^{1,1}([-\tau, 0], X) \times Y$ . Then, under (A.1) – (A.4), for all  $0 < t < T$ , there exists a constant  $\zeta(r, T)$  such that

$$\|n(t) - m(t)\|_X \leq Me^{\alpha t} \left( \|\hat{n} - \hat{m}\|_E + \|\hat{n}_0 - \hat{m}_0\|_X \right) (1 + t\zeta(r, T)),$$

with  $r > \max\{\|(\hat{n}, \hat{n}_0)\|_{\mathbb{Y}}, \|(\hat{m}, \hat{m}_0)\|_{\mathbb{Y}}\}$ , where  $T$  is a common local existence time of  $n$  and  $m$ .

**Theorem 4.2** If the initial condition  $\hat{n} \geq 0$ , then under (A.1) – (A.4), the solution of the problem (2.2) is non-negative for any  $t \in [-\tau, T]$ , where  $T$  is the local existence time of the solution.

## 5 Global Existence of Solution

We denote by  $\varphi(t; t_0, x_0)$  the characteristic curve passing through  $(x_0, t_0) \in (0, +\infty) \times [0, T]$ , i.e., it is the solution of

$$\frac{\partial \varphi}{\partial t} = \gamma(\varphi(t; t_0, x_0), N[n(t)](\varphi(t; t_0, x_0))), \varphi(t_0; t_0, x_0) = x_0. \quad (5.1)$$

From (1.1), we have that for  $t \geq 0$ ,

$$n(x, t) = \begin{cases} \int_{\eta}^t e^{-\int_{\zeta}^t \mu(\varphi(s; t, x), N[n(s)](\varphi(s; t, x))) ds} \\ \quad \int_{-\tau}^0 R[n(\zeta + \sigma)](\varphi(\zeta; t, x)) \varphi_x(\zeta; t, x) d\zeta, & x < z(t), \\ \hat{n}_0(\varphi(0; t, x)) e^{-\int_0^t \mu(\varphi(s; t, x), N[n(s)](\varphi(s; t, x))) ds} \varphi_x(0; t, x) \\ \quad + \int_0^t e^{-\int_{\zeta}^t \mu(\varphi(s; t, x), N[n(s)](\varphi(s; t, x))) ds} \\ \quad \int_{-\tau}^0 R[n(\zeta + \sigma)](\varphi(\zeta; t, x)) \varphi_x(\zeta; t, x) d\zeta, & x > z(t), \end{cases} \quad (5.2)$$

where  $\eta$  is implicitly given by  $\varphi(\eta; t, x) = 0$ ,  $z(t) := \varphi(t; 0, 0)$  is the characteristic curve coming from the origin, and

$$\varphi_{x_0}(t; t_0, x_0) := \exp \left( \int_{t_0}^t D\gamma(\varphi(s; t_0, x_0), N[n(s)](\varphi(s; t_0, x_0))) ds \right).$$

In order to obtain global existence of solution, we make the additional assumption :

$$(A.5) \text{ For any positive integrable function } n, \frac{\partial \gamma(x, N)}{\partial N} \frac{\partial N[n](x)}{\partial x} \geq 0.$$

**Lemma 5.1** *Let us assume the hypotheses (A.1) – (A.5) and let  $\mathbf{U}$  be a positive solution of the problem (2.5) up to time  $T$ . Then  $\|\mathbf{U}\|_{\mathbb{Y}}$  is upper bounded by a positive continuous function of  $t$  for all  $t \in [0, T]$ .*

*Proof.* Since

$$\|\mathbf{U}(t)\|_{\mathbb{Y}} = \|n(t)\|_X + \|n_x(t)\|_X + \|n_t\|_E + \left\| \frac{\partial n(t + \sigma)}{\partial \sigma} \right\|_E, \quad (5.3)$$

we step by step obtain the estimates of each term.

**The estimate of  $\|n(t)\|_X$ :** From (5.2), we have that

$$\frac{d\|n(t)\|_X}{dt} = \int_0^\infty \int_{-\tau}^0 R[n(t + \sigma, y)](s) d\sigma - \mu(s, N[n(t)](s)) n(s, t) ds.$$

From the positivity of  $n(t)$  and  $\mu(x, N)$ , we have that

$$\begin{aligned}
\|n(t)\|_X &\leq \int_0^t \int_0^\infty \int_{-\tau}^0 R[n(r+\sigma)](s) d\sigma ds dr + \|\hat{n}_0\|_X \\
&\leq \bar{R} \int_0^t \int_0^\infty \int_{-\tau}^0 n(r+\sigma, y) d\sigma dy dr + \|\hat{n}_0\|_X \\
&= \bar{R} \int_0^\infty \int_0^t \int_{-\tau}^0 n(r+\sigma, y) dr d\sigma dy + \|\hat{n}_0\|_X \\
&= \bar{R} \int_0^\infty \int_{-\tau}^0 \int_\sigma^{t+\sigma} n(\xi, y) d\xi d\sigma dy + \|\hat{n}_0\|_X \\
&\leq \bar{R}\tau \int_0^t \|n(\xi)\|_X d\xi + \bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X,
\end{aligned}$$

where  $\bar{R} = \max\{R_0, R_1, R_2\}$ . Using the Gronwall's lemma, we have that

$$\|n(t)\|_X \leq (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t}. \quad (5.4)$$

**The estimate of  $\|n_t\|_E$ :** We have that if  $t - \tau \geq 0$ , then

$$\begin{aligned}
\|n_t\|_E &= \int_{-\tau}^0 \int_0^\infty n(t+\sigma, y) d\sigma dy \\
&\leq \int_{-\tau}^0 (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau(t+\sigma)} d\sigma \\
&\leq \tau (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t},
\end{aligned}$$

otherwise, i.e.  $t - \tau < 0$ , we have

$$\begin{aligned}
\|n_t\|_E &= \int_0^\infty \int_{-\tau}^{-t} n(t+\sigma, y) d\sigma dy + \int_0^\infty \int_{-t}^0 n(t+\sigma, y) d\sigma dy \\
&\leq \int_0^\infty \int_{-\tau}^{-t} n(\xi, y) d\xi dy + \int_0^\infty \int_0^t n(\xi, y) d\xi dy \\
&\leq \|\hat{n}\|_E + t(\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t}.
\end{aligned}$$

Then

$$\|n_t\|_E \leq \|\hat{n}\|_E + (t + \tau)(\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t}. \quad (5.5)$$

**The estimate of  $\|n_x(t)\|_X$ :** Let  $\tilde{n}(t) := n(\varphi(t; t_0, x_0), t)$ . We first give the estimate of  $\|\tilde{n}(t)\|_\infty$ . Let  $\tilde{n}(t) := n(\varphi(t; t_0, x_0), t)$ . We consider (1.1)<sub>1</sub> as a directional derivative and, so for a fixed  $(x_0, t_0)$ , we have that

$$\frac{d\tilde{n}(t)}{dt} = \int_{-\tau}^0 R[n(t+\sigma)] \varphi(t; t_0, x_0) d\sigma - \lambda(\varphi(t; t_0, x_0), t) \tilde{n}(t) dt + \tilde{n}(0),$$

where  $\lambda(x, t) = D\gamma(x, N[n(t)(x)]) + \mu(s, N[n(t)](x))$ . Then

$$\tilde{n}(t) = \frac{d\tilde{n}(t)}{dt} = \int_0^t \int_{-\tau}^0 R[n(r+\sigma)] \varphi(r; t_0, x_0) d\sigma dr - \int_0^t \lambda(\varphi(r; t_0, x_0), t) \tilde{n}(r) dr + \tilde{n}(0),$$

Recall that  $\frac{\partial \gamma(x, N)}{\partial N} \frac{\partial N[n](x)}{\partial x}$  and  $\mu(x, N)$  are non-negative. We denote  $\lambda_0 = \inf_{x,t} \{\gamma_x(x, N(x, t))\}$  and assume that  $\lambda_0$  is negative, and  $\lambda_0 = 0$  otherwise. Then

$$\begin{aligned}
\tilde{n}(t) &\leq \bar{R} \int_0^t \int_0^\infty \int_{-\tau}^0 n(r + \sigma, y) d\sigma dy dr - \lambda_0 \int_0^t \tilde{n}(r) dr + \tilde{n}(0) \\
&\leq \bar{R} \int_0^\infty \int_{-\tau}^0 \int_\sigma^{t+\sigma} n(\xi, y) d\xi d\sigma dy - \lambda_0 \int_0^t \tilde{n}(r) dr + \tilde{n}(0) \\
&\leq \bar{R}\tau \|\hat{n}\|_E + \bar{R}\tau \int_0^t \|n(\xi)\|_X d\xi - \lambda_0 \int_0^t \tilde{n}(r) dr + \tilde{n}(0) \\
&\leq \bar{R}\tau \|\hat{n}\|_E + \tilde{n}(0) + (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} - \lambda_0 \int_0^t \tilde{n}(r) dr.
\end{aligned}$$

Using the Gronwall's lemma, we have that if  $\lambda_0 \neq -\bar{R}\tau$ , then

$$\|\tilde{n}(t)\|_\infty \leq (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_\infty) (e^{-\lambda_0 t} + 1) + (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} \left(1 - \frac{\lambda_0}{\bar{R}\tau + \lambda_0}\right) =: g_1(t), \quad (5.6)$$

otherwise, i.e.  $\lambda_0 = -\bar{R}\tau < 0$ , we have

$$\|\tilde{n}(t)\|_\infty \leq (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_\infty) (e^{-\lambda_0 t} + 1) + (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) (e^{\bar{R}\tau t} - \lambda_0 t e^{-\lambda_0 t}) =: g_2(t). \quad (5.7)$$

Let us consider  $n$  as a function of  $t$  and  $\xi := \varphi(0; t, x)$  for  $x > z(t)$ , and as a function of  $t$  and  $\eta$ , with  $\eta$  given by  $\varphi(\eta; t, x) = 0$  for  $x < z(t)$ , i.e.

$$n(x, t) = \bar{n}(t, \eta(x, t)), 0 < x < z(t), n(x, t) = \bar{\bar{n}}(t, \xi(x, t)), x > z(t).$$

Hence, we have that

$$n_x(x, t) = \begin{cases} \bar{n}_\eta(t, \eta) \partial_x \eta(x, t), & x < z(t), \\ \bar{\bar{n}}_\xi(t, \xi) \partial_x \xi(x, t), & x > z(t), \end{cases}$$

and

$$\begin{aligned}
\|n_x(t)\|_X &= \int_0^{z(t)} |n_x(x, t)| dx + \int_{z(t)}^\infty |n_x(x, t)| dx \\
&\leq \int_0^t |\bar{n}_\eta(t, \eta)| d\eta + \int_0^\infty |\bar{\bar{n}}_\xi(t, \xi)| d\xi.
\end{aligned}$$

The expression of  $\bar{n}_\eta$  and  $\bar{\bar{n}}_\xi$  are similar as  $\bar{u}_\tau$  and  $\bar{\bar{u}}_\xi$  in the proof of Lemma 1 in [1], we obtain the following result by using the same method:

$$\|n_x(t)\|_X \leq f_1(t) + [\|\hat{n}_0\|_\infty e^{-\lambda_0 t} + g(t)] (\gamma_2^0 \int_0^t \|(N_x(s))^2\|_X ds + \gamma_1^0 \int_0^t \|N_{xx}(s)\|_X ds). \quad (5.8)$$

where  $f_1(t)$  is a positive and increasing function which is similar as the one in (6.8) of [1],  $g(t) := g_1(t) + g_2(t)$ . Moreover, under (A.1) and (A.2) and using the  $\|N_x\|_{L^1} \leq c' \|n(t)\|_{L^1}$  and  $\|N_x\|_\infty \leq \|N_x\|_{W^{1,1}} \leq c'' \|n(t)\|_{W^{1,1}}$ , the last two integrals of (5.8) can be bounded as follows

$$\begin{aligned} \int_0^t \|(N_x(s))^2\|_X ds &\leq \int_0^t \|N_x(s)\|_\infty \|N_x(s)\|_X ds \\ &\leq c'(\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} \int_0^t \|N_x(s)\|_\infty ds \\ &\leq c'(\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} c'' \int_0^t (\|n(s)\|_X + \|n_x(s)\|_X) ds \end{aligned} \quad (5.9)$$

and

$$\int_0^t \|N_{xx}(s)\|_X ds \leq c'' \int_0^t (\|n(s)\|_X + \|n_x(s)\|_X) ds. \quad (5.10)$$

From (5.8)-(5.10), we have that

$$\|n_x(t)\|_X \leq f(t) + f_2(t) \int_0^t \|n(r)\|_X dr, \quad (5.11)$$

where  $f_2(t) = c''[\|\hat{n}_0\|_\infty e^{-\lambda_0 t} + g(t)](\gamma_2^0 c'(\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} + \gamma_1^0)$  and  $f(t) = f_1(t) + t f_2(t) \bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X e^{\bar{R}\tau t}$ .

From the Gronwall's lemma, we have that

$$\|n_x(t)\|_X \leq f(t) + f_2(t) \int_0^t f(s) \exp\left(\int_s^t f_2(\zeta) d\zeta\right) ds. \quad (5.12)$$

Then, we have that  $\|n_x(t)\|_X \leq H_1(t, \|\mathbf{U}_0\|_{\mathbb{Y}})$ , where  $H_1(t, \|\mathbf{U}_0\|_{\mathbb{Y}})$  denotes the R.H.S of (5.13) with the  $\|\hat{n}\|_E$ ,  $\|\hat{n}_0\|_\infty$  and  $\|\hat{n}_0\|_{L^1}$  replaced by  $\|\mathbf{U}_0\|_{\mathbb{Y}}$ .

**The estimate of  $\left\|\frac{\partial n(t+\sigma)}{\partial \sigma}\right\|_E$ :**

$$\int_0^\infty \int_{-\tau}^0 \left| \frac{\partial n(t+\sigma, x)}{\partial \sigma} \right| d\sigma dx = \int_0^\infty \int_{-\tau+t}^t \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx.$$

If  $t > \tau$ ,  $n(\xi, x)$  satisfies (1.1)<sub>1</sub>, then we have that

$$\begin{aligned} &\int_0^\infty \int_{-\tau+t}^t \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx \\ &\leq \int_0^\infty \int_0^t \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx \\ &\leq \gamma^0 \int_0^\infty \int_0^\infty \left| \frac{\partial n(\xi, x)}{\partial x} \right| dx d\xi + (\mu^0 + \gamma_1^0) \int_0^t \int_0^\infty |n(\xi, x)| dx d\xi \\ &\quad + \int_0^t \int_0^\infty \int_{-\tau}^0 R[n(\xi + \sigma)](x) d\sigma dx d\xi \end{aligned}$$



$$\begin{aligned}
&\leq \gamma^0 t N_2(t, \|\mathbf{U}_0\|_{\mathbb{Y}}) + (\mu^0 + \gamma_1^0) t (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} + \bar{R} \int_0^t \int_0^\infty \int_{-\tau}^0 n(\xi + \sigma, y) d\sigma dy d\xi \\
&\leq \gamma^0 t H_1(t, \|\mathbf{U}_0\|_{\mathbb{Y}}) + (\mu^0 + \gamma_1^0) t (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} \\
&\quad + \bar{R} t (\|\hat{n}\|_E + (t + \tau) (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t}),
\end{aligned}$$

otherwise, i.e.  $t \leq \tau$ , we have

$$\begin{aligned}
&\int_0^\infty \int_{-\tau+t}^t \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx \\
&\leq \int_0^\infty \int_{-\tau+t}^0 \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx + \int_0^\infty \int_0^t \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx \\
&\leq \int_0^\infty \int_{-\tau}^0 \left| \frac{\partial \hat{n}(\xi, x)}{\partial \xi} \right| d\xi dx + \int_0^\infty \int_0^t \left| \frac{\partial n(\xi, x)}{\partial \xi} \right| d\xi dx \\
&\leq \|\mathbf{U}_0\|_{\mathbb{Y}} + \gamma^0 t H_1(t, \|\mathbf{U}_0\|_{\mathbb{Y}}) + (\mu^0 + \gamma_1^0) t (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} \\
&\quad + \bar{R} t (\|\hat{n}\|_E + (t + \tau) (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t}).
\end{aligned}$$

Then

$$\left\| \frac{\partial n(t + \sigma)}{\partial \sigma} \right\|_E \leq \|\mathbf{U}_0\|_{\mathbb{Y}} + \gamma^0 t H_1(t, \|\mathbf{U}_0\|_{\mathbb{Y}}) + (\mu^0 + \gamma_1^0) t (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t} + \bar{R} t (\|\hat{n}\|_E + (t + \tau) (\bar{R}\tau \|\hat{n}\|_E + \|\hat{n}_0\|_X) e^{\bar{R}\tau t}). \quad (5.13)$$

From (5.3), (5.4), (5.5), (5.12) and (5.13), we have the proof of this theorem.  $\square$

Using the same method in the Section 6.3 of [1], we have the following result:

**Lemma 5.2** *Under (A.1) – (A.5) and for any  $T^* > 0$ , the problem (2.5) has a unique solution up to  $T^*$ , which is positive whenever the initial condition  $\mathbf{U}_0 > 0$ .*

By Lemma 2.1, we have the following result:

**Theorem 5.3** *Under (A.1) – (A.5) and for any  $T^* > 0$ , the problem (2.2) has a unique solution up to  $T^*$ , which is positive whenever the initial condition  $\hat{n} > 0$ .*

## 6 Remark

In this section, we give the typical examples of the operators  $N$  and  $R$ . The environment experienced by an individual of size  $x$  when the population density is  $n(x, t)$  can be given by

$$N[n(t)](x) = \int_0^\infty \rho(x, y) n(t, y) dy.$$

Many examples of such a sort of environments with different  $\rho$  can be seen, for instance, in the references [1], [16] [17], [18], [19], [20] and [21]. A simple example of the operator  $R$  can be given by

$$R[n(t + \sigma)](x) = \int_0^\infty \beta(\sigma, x, y) n(t + \sigma, y) d\sigma dy$$

where  $\beta(\sigma, x, y)$  denotes the reproduction rate which individuals of size  $y$  give birth to the individuals of size  $x$  after a time lag  $-\sigma$  starting from conception.  $\beta(\sigma, x, y)$  is supposed to satisfy the following conditions:

(H.1)  $\beta \in C([-\tau, 0] \times [0, \infty) \times [0, \infty))$ ,  $\beta \geq 0$  and is uniformly bounded by  $R_2$ .

(H.2) For all  $\sigma, y \in [-\tau, 0] \times [0, \infty)$ ,  $\int_0^\infty \beta(\sigma, x, y)dx \leq R_0$ ,  $\beta(\sigma, \cdot, y) \in C^1[0, \infty)$  and  $\int_0^\infty \beta_x(\sigma, x, y)dx \leq R_1$ .

In this paper we consider the case in which the process of the recruitment is influenced by the environment from start to finish, hence we assume that the rate  $\beta$  is also dependent on the total environment in the time lag between the beginning and the end of the recruitment. Under such an assumption, the operator  $R$  can be given by

$$R[n(t + \sigma)](x) = \int_0^\infty \beta(\sigma, x, y, \mathcal{N}(\sigma, x))n(t + \sigma, y)d\sigma dy,$$

where

$$\mathcal{N}(\sigma, x) = \int_{-\tau}^0 \chi(\sigma, \xi)N[n(t + \xi)](x)d\xi$$

and

$$\chi(\sigma, \xi) = \begin{cases} 1, & \xi \geq \sigma, \\ 0, & \xi < \sigma. \end{cases}$$

$\beta(\sigma, x, y, \mathcal{N})$  is supposed to satisfy the following conditions:

(H.3)  $\beta \in C([-\tau, 0] \times [0, \infty) \times [0, \infty) \times [0, \infty))$ ,  $\beta \geq 0$  and is uniformly bounded by  $R_2$ .

(H.4) For all  $\sigma, y, \mathcal{N} \in [-\tau, 0] \times [0, \infty) \times [0, \infty)$ ,  $\int_0^\infty \beta(\sigma, x, y, \mathcal{N})dx \leq R_0$ ,  $\beta(\sigma, \cdot, y, \mathcal{N}) \in C^1[0, \infty)$  and  $\int_0^\infty \beta_x(\sigma, x, y, \mathcal{N})dx \leq R_1$ . Moreover,  $\int_0^\infty \beta(\sigma, x, y, \mathcal{N})dx$  and  $\int_0^\infty \beta_x(\sigma, x, y, \mathcal{N})dx$  are Lipschitzian functions with respect to  $\mathcal{N}$ .

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